Head-on collision of gravitational plane waves with noncollinear polarization: A new class of analytic models

Dimitri Tsoubelis
Department of Mathematics, University of Patras, 261 10 Patras, Greece

Anzhong Wang[a)]
Department of Physics, Division of Theoretical Physics, University of Ioannina, P.O. Box 1186, GR. 451 10 Ioannina, Greece

(Received 15 July 1991; accepted for publication 10 October 1991)

A new four-parameter class of exact solutions of Einstein's field equations is obtained, using the inverse scattering method of Belinsky and Zakharov. Its members represent the head-on collision of a variably polarized gravitational plane wave with one having constant polarization and, in general, different profile, or with an infinitely thin shell of null dust. In some of these models no curvature singularity develops along the future boundary of the region of interaction. In certain cases the singularity avoidance is the direct result of the noncollinear polarization of the waves involved in the collision.

I. INTRODUCTION

Analytic, or exact, solutions of Einstein's equations that can be interpreted as models of the head-on collision of a pair of gravitational plane waves have great theoretical value in the context of general relativity. Such models, however, had been very hard to obtain before the appearance of the solution generating techniques, such as the inverse scattering or soliton method of Belinsky and Zakharov (BZ),[1,2] which were invented quite recently. These techniques have made possible the construction of several new families of colliding waves models exhibiting an ever increasing variety of physical characteristics and global behavior, compared with the first solutions found by Szekeres[3,4] and Khan and Penrose[5] in the early seventies (for a recent concise review of the subject, see Ref. 6).

In this vein, the BZ technique is used in the present paper for deriving a new class of colliding gravitational waves solutions with the following features. The members of the family are distinguished by a tetrad of real parameters, (a, δ_1, δ_2, q), the first three of which determine the shape of each of the two wave pulses involved in the collision. The last of the above parameters, on the other hand, determines the polarization of one of the incoming waves relative to a fixed set of coordinate axes. Thus, by tuning the parameter tetrad appropriately, one can cover a large variety of physically distinct cases, from the collision of two gravitational plane waves with different profile and polarization to the interaction of a variably polarized gravitational wave with an infinitely thin shell of null dust. Moreover, the polarization parameter q can approach the value $q=0$ continuously and figures in the metric coefficients in a manner that makes it easy to compare the noncollinear ($q\neq0$) models with their collinear ($q=0$) counterparts.

It must be pointed out that several of the above features are also shared by the solutions obtained recently by Ferrari–Ibanez–Bruni,[7,8] Ernst–Garcia–Hauser,[9–11] Li–Hauser–Ernst[12,13] and Feinstein–Senovilla.[14] However, the class of solutions presented in this paper differs from the above in several respects. To be specific, we compare them here briefly with the Ferrari–Ibanez–Bruni[7,8] (FIB) models, the only family of solutions among those mentioned above that was also constructed using the BZ technique. To begin with, the FIB metrics involve two parameters less than ours; they are, essentially, the two-soliton counterpart of our $\delta_1 = \delta_2 = 1/2$ subclass of solutions. As a result, the FIB models cannot cover the case of collisions in which shells of null dust are present. Moreover, due to their construction by a pair of solitons placed symmetrically on the propagation axis, the FIB models retain a symmetry between the left and right moving waves. In contrast, our solutions are the product of a one-soliton "perturbation" of an initially diagonal metric. As a result, in the generic case, the two legs of the incident radiation differ from each other completely (for further comments, see Ref. 15 and following sections).

The structure of the paper is the following. In Sec. II, we present a short outline of the BZ soliton technique and introduce our notation. In Sec. III, a one-soliton class of new solutions is derived by applying the BZ method to the three-parameter class of diagonal solutions obtained by the present authors recently.[16,17] The extension of the above solutions, which renders their interpretation as colliding gravitational waves possible, is presented in Sec. IV. Section V is devoted to a detailed analysis of some

[a)]Permanent address: Physics Dept., Northeast Normal University, Changchun, Jilin, Peoples Republic of China.

 0022-2488/92/031054-11$03.00

representive members of the family of solutions obtained in the previous sections, in terms of the corresponding Weyl or conformal curvature tensor. The paper closes with Sec. VI, in which the singularity behavior of the whole four-parameter class of solutions is considered.

II. THE METRIC FORM AND SOLUTION TECHNIQUE

In this section, we shall present a brief outline of the BZ technique of solving the Einstein vacuum equations when the space-time metric admits a pair of commuting Killing vector fields. This only serves the purpose of introducing the notation employed in the following sections. For details on the BZ soliton technique, the reader may consult Refs. 1, 2, and 15.

When one considers solutions of the field equations that could represent colliding gravitational plane waves, it is convenient to write the space-time metric in the form

$$ds^2=f(t,z)(dt^2-dz^2)-g_{ab}(t,z)dx^a\,dx^b, \tag{2.1}$$

where a, b run from 2 to 3. Furthermore, one can set

$$\det(g)\equiv\det(g_{ab})=t^2, \tag{2.2}$$

without loss of generality.

Suppose, now, that $f^{(0)}$, $g^{(0)}{}_{ab}$ is a known solution of the Einstein vacuum field equations. In the context of the BZ "soliton" or "inverse scattering method," such a solution is referred to as the "seed metric" from which a new solution f, g_{ab} is produced via the following steps. First, one chooses the number N of the "poles" and the associated "pole trajectories"

$$\mu_k(t,z)=w_k-z\pm\sqrt{(w_k-z)^2-t^2}, \tag{2.3}$$

where $k=1,\ldots,N$ and the w_k's are arbitrary real or complex constants. The natural number N specifies the soliton index of the new solutions relative to the seed metric.

The next and crucial step in the BZ technique consists of integrating the "Schrödinger equations"

$$\left(\frac{\partial}{\partial t}+\frac{2\lambda t}{t^2-\lambda^2}\frac{\partial}{\partial\lambda}\right)\Psi=\frac{t\Lambda+\lambda\Omega}{t^2-\lambda^2}\Psi, \tag{2.4a}$$

$$\left(\frac{\partial}{\partial z}+\frac{2\lambda^2}{t^2-\lambda^2}\frac{\partial}{\partial\lambda}\right)\Psi=\frac{t\Omega+\lambda\Lambda}{t^2-\lambda^2}\Psi \tag{2.4b}$$

for the 2×2 matrix, or "wave function," $\Psi(t,z;\lambda)$, subject to the "initial condition"

$$\lim_{\lambda\to0}\Psi(t,z;\lambda)=g^{(0)}(t,z). \tag{2.5}$$

In Eqs. (2.4)

$$\Lambda\equiv tg^{(0)}{}_{,t}g^{(0)-1}\quad\text{and}\quad\Omega\equiv tg^{(0)}{}_{,z}g^{(0)-1}, \tag{2.6}$$

where $(\ )_{,x}$ denotes partial differentiation with respect to x.

When $g^{(0)}$ is diagonal, as in the case to be treated in the following sections, the integration of Eqs. (2.4) is simplified by assuming that Ψ is diagonal, too. It is also found to be convenient to write the metric given by Eqs. (2.1) and (2.2) in the form

$$ds^2=f(t,z)(dt^2-dz^2)-t[\chi(dx^2)^2+\chi^{-1}(dx^3-q_2\,dx^2)^2], \tag{2.7}$$

and to introduce the coordinates $\eta\in(0,1)$, $\mu\in(-1,1)$ via the relations

$$t=(\Delta\delta)^{1/2},\quad\Delta\equiv1-\eta^2,\quad\delta\equiv1-\mu^2, \tag{2.8a}$$

and

$$z=\eta\mu. \tag{2.8b}$$

Then, Ψ can be written in the form

$$\Psi(\eta,\mu;\lambda)=(2\lambda w)^{1/2}\,\mathrm{diag}(\Sigma,\Sigma^{-1}), \tag{2.9}$$

whereby Eqs. (2.4) and (2.5) become

$$\left(\Delta\frac{\partial}{\partial\eta}+\lambda\frac{\partial}{\partial\mu}-2\lambda\eta\frac{\partial}{\partial\lambda}\right)\ln\Sigma=\Delta\,(\ln\chi^{(0)})_{,\eta}, \tag{2.10a}$$

$$\left(\delta\frac{\partial}{\partial\mu}+\lambda\frac{\partial}{\partial\eta}-2\lambda\mu\frac{\partial}{\partial\lambda}\right)\ln\Sigma=\delta\,(\ln\chi^{(0)})_{,\mu}, \tag{2.10b}$$

and

$$\lim_{\lambda\to0}\Sigma(\eta,\mu;\lambda)=\chi^{(0)}(\eta,\mu), \tag{2.11}$$

respectively. Let us, here, note that the integrability condition of the first-order system of Eqs. (2.10) is provided by the field equation[16]

$$[\Delta\,(\ln\chi^{(0)})_{,\eta}]_{,\eta}-[\delta\,(\ln\chi^{(0)})_{,\mu}]_{,\mu}=0. \tag{2.12}$$

The remaining steps in the BZ solution-generating technique are all algebraic in nature, but quite complicated. However, a very convenient form of the final formulas has been obtained by Economou and Tsoubelis[15] recently. Those which are pertinent to the case under study will be explicitly quoted in the next section.

III. ONE-SOLITON SOLUTIONS

In Refs. 16 and 17, it was shown that one solution of the vacuum field equations is provided by the metric (2.7), if the coefficients, χ, q_2, and f are chosen to be equal to $\chi^{(0)}$, $q_2{}^{(0)}$, and $f^{(0)}$, respectively, where

$$\chi^{(0)}=(\Delta\delta)^a\left(\frac{1-\eta}{1+\eta}\right)^{\delta_1}\left(\frac{1-\mu}{1+\mu}\right)^{\delta_2}, \qquad (3.1a)$$

$$q_2{}^{(0)}=0, \qquad (3.1b)$$

and

$$f^{(0)}=C\frac{(1-\eta)^{b_1}(1+\eta)^{c_1}(1-\mu)^{b_2}(1+\mu)^{c_2}}{(\eta+\mu)^{\delta_+^2}(\eta-\mu)^{\delta_-^2}}. \qquad (3.1c)$$

Here, (a, δ_1, δ_2) is a triad of real parameters that characterize the solution, C is an arbitrary constant, and

$$\delta_\pm\equiv\delta_1\pm\delta_2,\quad b_A\equiv(a+\delta_A)^2-\tfrac{1}{4},\quad c_A\equiv(a-\delta_A)^2-\tfrac{1}{4}, \qquad (3.2)$$

with $A=1,2$.

Considering the above metric as the seed, a one-soliton solution can be generated by the method described in the last section as follows. In accordance with Eq. (2.3), let us choose the single-pole trajectory to read

$$\mu_1=1-z-\sqrt{(1-z)^2-t^2}=(1-\eta)(1-\mu). \qquad (3.3)$$

Furthermore, let us introduce the functions

$$\Sigma(\eta,\mu)\equiv\Sigma(\eta,\mu;\mu_1),\quad \sigma_1(\eta,\mu)\equiv\mu_1/(\Delta\delta)^{1/2},$$

and

$$s_1(\eta,\mu)\equiv Q_1\chi^{(0)}/\Sigma_1{}^2, \qquad (3.4)$$

where Q_1 is an arbitrary real constant. Then, the Economou–Tsoubelis[15] formulas mentioned above give the metric coefficients for the new solution in the form

$$\chi=\{|\sigma_1|(1+s_1{}^2)/[1+(\sigma_1 s_1)^2]\}\chi^{(0)}, \qquad (3.5a)$$

$$q_2=\{s_1(\sigma_1{}^2-1)/[1+(\sigma_1 s_1)^2]\}\chi^{(0)}, \qquad (3.5b)$$

$$f=C_{ph}[\sigma_1(1+s_1)^2/(\Delta\delta)^{1/4}s_1(\sigma_1{}^2-1)]f^{(0)}, \qquad (3.5c)$$

where the constant C_{ph} is arbitrary.

It is now obvious that the new metric is completely specified as soon as the solution $\Sigma(\eta,\mu;\lambda)$ of Eqs. (2.10) and (2.11) is obtained. In the present case, this solution is easy to obtain, but involves tedious calculations. Therefore, we quote the final result which reads

$$\Sigma=\frac{(2\lambda w)^{a+\delta_1}}{[(1+\eta)(1+\mu)+\lambda]^{\delta_+}[(1+\eta)(1-\mu)-\lambda]^{\delta_-}}. \qquad (3.6)$$

In conclusion, a new family of solutions of the Einstein vacuum equations, which is characterized by the parameters $(a, \delta_1, \delta_2, q)$, is given by combining Eqs. (3.3)–(3.6) and is expressed by the formulas

$$\chi=[(1-\eta^2)(1-\mu^2)]^{1/2}(A/B)\chi^{(0)}, \qquad (3.7a)$$

$$q_2=-2q(\eta-\mu)^{2\delta_-+1}(\chi^{(0)})^2/B, \qquad (3.7b)$$

$$f=C_{ph}\frac{(1-\eta)^{b'_1}(1+\eta)^{c'_1}(1-\mu)^{b'_2}(1+\mu)^{c'_2}}{(\eta+\mu)^{\delta_+^2}(\eta-\mu)^{(\delta_-+1)^2}}A, \qquad (3.7c)$$

where $\chi^{(0)}$ is given by Eq. (3.1a),

$$A\equiv(1-\eta)^{2(a+\delta_1)}(1+\mu)^{2(a-\delta_2)} + q^2(\eta-\mu)^{4\delta_-}(1+\eta)^{2(a-\delta_1)}(1-\mu)^{2(a+\delta_2)}, \qquad (3.8a)$$

$$B\equiv(1-\eta)^{2(a+\delta_1)}(1+\mu)^{2(a-\delta_2)}(1+\eta)(1-\mu) + q^2(\eta-\mu)^{4\delta_-}(1+\eta)^{2(a-\delta_1)}(1-\mu)^{2(a+\delta_2)} \times(1-\eta)(1+\mu), \qquad (3.8b)$$

$$b'_A\equiv(a+\delta_A)(a+\delta_A-1),$$

$$c'_A\equiv(a-\delta_A)(a-\delta_A-1), \qquad (3.9)$$

and the free parameter $q=4^{\delta_1-a}Q_1$.

In the following section the above family of solutions is used for the construction of colliding gravitational waves models.

IV. COLLIDING GRAVITATIONAL WAVES

Let us now introduce the pair of null coordinates (u, v) related to (t, z) and (η, μ) by

$$t=1-u^{2n}-v^{2m},\quad z=u^{2n}-v^{2m}, \qquad (4.1)$$

and

$$\eta=u^nR+v^mS,\quad \mu=u^nR-v^mS, \qquad (4.2a)$$

where

$$R \equiv \sqrt{1-v^{2m}}, \quad S \equiv \sqrt{1-u^{2n}}, \tag{4.2b}$$

respectively, with the parameter pair (n, m) determined by (δ_1, δ_2) via

$$n = 1/[2-\delta_+{}^2], \quad m = 1/[2-(\delta_- + 1)^2]. \tag{4.3}$$

In terms of u, v the line element (2.7) becomes

$$ds^2 = 2e^{-M}\,du\,dv - e^{-U} \times [\chi(dx^2)^2 + \chi^{-1}(dx^3 - q_2\,dx^2)^2], \tag{4.4}$$

where

$$e^{-M} = 8nmu^{2n-1}v^{2m-1}f(u,v), \tag{4.5a}$$

$$e^{-U} = t(u,v). \tag{4.5b}$$

This form of the metric will be the basis of our discussion from this point on.

Specifically, let us call the coordinates appearing in Eq. (4.4) $\{x^\mu\}$, where $\mu = 0, 1, 2, 3$ and $(x^1, x^2) \equiv (u,v)$. Using this notation, we can distinguish the following regions of space-time: $R_{\rm I} = \{x^\mu : u < 0, v < 0\}$, $R_{\rm II} = \{x^\mu : u < 0, 0 < v < 1\}$, $R_{\rm III} = \{x^\mu : 0 < u < 1, v < 0\}$, and $R_{\rm IV} = \{x^\mu : u > 0, v > 0, u^{2n} + v^{2m} < 1\}$. We then assume that the solution obtained in the last section holds only in $R_{\rm IV}$, which will be referred to, hereafter, as the region of interaction, and extend it across the null hypersurfaces $u = 0$, and $v = 0$ in the following manner. First, we restrict the range of values of the parameter pair (m, n) by the condition

$$m \in [1, \infty), \quad n \in \{\tfrac{1}{2}\} \cup [1, \infty). \tag{4.6}$$

Then, we apply the Khan–Penrose substitution[5]

$$F(u,v) \to F(\tilde{u},\tilde{v}) \equiv F(uH(u), vH(v)), \tag{4.7}$$

where F is any of the functions appearing in the metric coefficients $g_{\mu\nu}(u,v)$ in $R_{\rm IV}$ and $H(x)$ is the Heaviside unit step function.

Along the lines described in detail in Refs. 16 and 17, it can be shown that the above method of extension guarantees that the Einstein vacuum equations are satisfied in all regions $R_{\rm I}$–$R_{\rm III}$ and the appropriate junction conditions are satisfied along the hypersurfaces $u = 0$ and $v = 0$. Moreover, the latter are empty, unless $n = 1/2$. In that case the separation hypersurface $u = 0$ is occupied by an "infinitely thin shell of null dust." More specifically, one finds that the stress-energy tensor, $T_{\mu\nu}$, corresponding to the extended solution is given by

$$\kappa T_{\mu\nu} = \frac{2nu^{2n-1}\delta(u)}{1 - v^{2m}H(v)}\,\delta^0_\mu \delta^0_\nu, \tag{4.8}$$

where $\kappa(\equiv 8\pi G/c^4)$ denotes the Einstein gravitational constant and $\delta(u)$ stands for Dirac's "delta function."

It is obvious that Eqs. (4.4) and (4.7) imply that

$$ds^2 = 2\,du\,dv - (dx^2)^2 - (dx^3)^2, \tag{4.9}$$

in $R_{\rm I}$, while $g_{\mu\nu} = g_{\mu\nu}(v)$ and $g_{\mu\nu} = g_{\mu\nu}(u)$ in $R_{\rm II}$ and $R_{\rm III}$, respectively. Combined with Eq. (4.8), this shows that the extended solution can be interpreted as representing the collision of a gravitational plane wave incident from the left, say, in the flat region $R_{\rm I}$ with a similar wave incident from the right. When $n = 1/2$, the leading edge of the latter is occupied by a shell of null dust. Moreover, by combining Eqs. (3.7b), (4.2), and (4.7), one easily finds that g_{23} vanishes identically in $R_{\rm III}$, while $g_{23}(v) \neq 0$ in $R_{\rm II}$, unless $q = 0$. This means that the wave incident from the right has constant polarization—one says that it is a (+)-type of wave relative to the x^2, x^3 axes. In the wave incident from the left, on the other hand, both, the (+) and the ($\times$), polarization modes are present, in the generic case, and their respective contribution changes as the wave propagates. Thus our $q \neq 0$ solutions represent the collision of a constantly polarized gravitational plane wave with an oppositely moving one, which has a different and, in general, variable polarization, except when $q = 0$. When $q = 0$, the colliding waves have parallel polarization. The latter, however, is the collinear case studied in Refs. 16 and 17, and, therefore, will not concern our discussion until Sec. VI.

That the above interpretation is valid is brought out clearly by the study of the Weyl curvature tensor to which we now turn.

Introducing the functions $V(u,v)$ and $W(u,v)$ via

$$V = \tfrac{1}{2}\ln(\chi^2 + q_2{}^2) \text{ and } \sinh W = q_2/\chi, \tag{4.10}$$

respectively, we find that the vectors

$$l^\mu = e^{M/2}\delta^\mu_v, \quad n^\mu = e^{M/2}\delta^\mu_u,$$
$$m^\mu = \zeta^2\delta^\mu_2 + \zeta^3\delta^\mu_3, \quad \bar{m}^\mu = \bar{\zeta}^2\delta^\mu_2 + \bar{\zeta}^3\delta^\mu_3, \tag{4.11}$$

where

$$\zeta^2 = \frac{e^{(U-V)/2}}{\sqrt{2}}\left\{\cosh\frac{W}{2} + i\sinh\frac{W}{2}\right\},$$
$$\zeta^3 = \frac{e^{(U+V)/2}}{\sqrt{2}}\left\{\sinh\frac{W}{2} + i\cosh\frac{W}{2}\right\}, \tag{4.12}$$

and the overbar denotes complex conjugation, form a null tetrad. The five "Weyl scalars" relative to such a tetrad completely characterize the Weyl curvature tensor. In the present case only three of these complex scalar functions survive. Denoting them by $\Psi_i(u,v)$, $i = 0, 2, 4$, and omitting an all-present, but irrelevant, scale factor equal to

$\exp(M)$, we quote Szekeres' formulas[4] for the Weyl scalars in $R_{\rm IV}$, correcting a printing mistake in Ref. 4, at the same time:

$$\Psi_0^{\rm IV} = -\tfrac{1}{2}\{V_{,vv}\cosh W + (M_{,v} - U_{,v})V_{,v}\cosh W + 2\sinh W V_{,v}W_{,v} - i[W_{,vv} + (M_{,v} - U_{,v})W_{,v} - \sinh W\cosh W V_{,v}^2]\}, \quad (4.13a)$$

$$\Psi_4^{\rm IV} = -\tfrac{1}{2}\{V_{,uu}\cosh W + (M_{,u} - U_{,u})V_{,u}\cosh W + 2\sinh W V_{,u}W_{,u} + i[W_{,uu} + (M_{,u} - U_{,u})W_{,u} - \sinh W\cosh W V_{,u}^2]\}, \quad (4.13b)$$

$$\Psi_2^{\rm IV} = \tfrac{1}{12}\{2(M_{,uv} - U_{,uv} + W_{,u}W_{,v} + \cosh^2 W V_{,u}V_{,v}) + i3\cosh W(V_{,u}W_{,v} - V_{,v}W_{,u})\}. \quad (4.13c)$$

Using these expressions and Eq. (4.7), on the other hand, one finds that the nonvanishing Weyl scalars in all regions $R_{\rm I}$–$R_{\rm IV}$ are given by

$$\Psi_0(u,v) = \Psi_0^{\rm IV}(\tilde{u},v)H(v) + \tfrac{1}{2}[V_{,v}(\tilde{u},0)\cosh W(\tilde{u},0) - iW_{,v}(\tilde{u},0)]\delta(v), \quad (4.14a)$$

$$\Psi_4(u,v) = \Psi_4^{\rm IV}(u,\tilde{v})H(u) - \tfrac{1}{2}[V_{,u}(0,\tilde{v})\cosh W(0,\tilde{v}) + iW_{,u}(0,\tilde{v})]\delta(u), \quad (4.14b)$$

$$\Psi_2(u,v) = \Psi_2^{\rm IV}(u,v)H(u)H(v). \quad (4.14c)$$

As shown by Szekeres,[18,19] Ψ_0 represents a transverse gravitational wave propagating in the direction specified by the null vector n^k, Ψ_4 represents a similar wave propagating in the direction of l^k, and Ψ_2 corresponds to the Coulomb-like component of the gravitational field. If this is combined with the fact that, as follows immediately from Eqs. (4.14), $\Psi_i \equiv 0$ in $R_{\rm I}$, $\Psi_0 = \Psi_0(v)$ is the only nonvanishing Weyl scalar in $R_{\rm II}$, and only $\Psi_4 = \Psi_4(u)$ is different from zero in $R_{\rm III}$, it becomes evident that the physical interpretation presented above is correct, in broad terms. For a more specific description of the type of waves involved in the collision, we first turn to the behavior of the Weyl scalars in the neighborhood of the leading wave fronts $u=0$ and $v=0$.

A lengthy but simple calculation leads to the expression

$$\Psi_4^{\rm III}(u) = ne^{2U}\{2an(4a^2-1)u^{4n-2} + 12a^2n\delta_+u^{3n-2} + 6a(2n-1)u^{2n-2} + (n-1)\delta_+u^{n-2}\}, \quad (4.15)$$

which, when combined with Eq. (4.14b), gives

$$\Psi_4^{\rm I\text{-}III} = \Psi_4^{\rm III}(0)H(u) + D(u)\delta(u) \equiv \Psi_4^{\rm III}(0)H(u) + n(\delta_+ + 2au^n)u^{n-1}\delta(u) \quad (4.16)$$

for the Weyl scalar Ψ_4 in the neighborhood of the hypersurface $u=0$ separating $R_{\rm III}$ from $R_{\rm I}$. Therefore, depending on the values of the parameters n and a, the wave incident from the right can be one of several types: smooth fronted $[\Psi^{\rm III}{}_4(0) = 0,\ D(u)=0]$, impulsive $[\Psi^{\rm III}{}_4(0) = 0,\ D(u)\neq 0]$, shock wave $[\Psi^{\rm III}{}_4(0)\neq 0,\ D(u) = 0]$, impulsive + shock, etc.

Similarly, one finds that the Weyl scalar Ψ_0 shows the following behavior in the neighborhood of the hypersurface $v=0$ separating $R_{\rm II}$ from $R_{\rm I}$.

When $m=1$, for example, then

$$\Psi_0^{\rm I\text{-}II} = [6a/(1+q^2)^2][(1-6q^2+q^4) - i4q(1-q^2)]H(v) + [1/(1+q^2)][(1-q^2) - i2q]\delta(v). \quad (4.17)$$

Thus, in this case, the gravitational wave incident from the left is of the shock + impulsive type, unless $a=0$. In the latter subcase only the impulsive component survives.

For $m=2$ we find that

$$\Psi_0^{\rm I\text{-}II} = 2(1+\delta_-)H(v), \quad (4.18)$$

which corresponds to a shock wave.

Finally, when $m>2$, the wave incident from the left is smooth fronted, because $\Psi_0^{\rm I\text{-}II}$ vanishes.

A more detailed analysis of the nature of the incident waves and the outcome of their interaction in $R_{\rm IV}$ would not be very illuminating if the four-parameter class of solutions obtained above were treated as a whole, because the corresponding expressions for the Ψ_i's are extremely complicated. Therefore, we proceed to study, in the next section, some representive subcases, separately.

V. SPECIFIC MODELS

In order to obtain a clearer physical picture of the class of colliding waves models constructed in the last section, we shall now turn to a more detailed analysis of some specific examples which belong to those subclasses of solutions for which the parameter pair (n, m) equals (1/2, 1), (1, 1), and (2, 1), respectively. This analysis will also reveal the relation between our solutions and those obtained recently by several other investigators.

A. Case A: $n=1/2, m=1$ ($\delta_1=\delta_2=0$)

As noted earlier, when $n=1/2$, the wave incident from the right is accompanied by an impulsive shell of null dust. Therefore, this physical feature characterizes all the models of the present case.

A-1: $a=0$. The Weyl scalars corresponding to this subcase are given by

$$\Psi_0(u,v)=[(1+q^2)S]^{-1}\{(1-q^2)-i2q\}\delta(v), \tag{5.1a}$$

$$\Psi_4(u,v)=\frac{v[(1-q^2)(1-v^2)+i2q(1+v^2)]}{2(1-v^2)[(1+q^2)^2(1-v^2)^2+16q^2v^2]}\times H(v)\delta(u), \tag{5.1b}$$

$$\Psi_2(u,v)=0. \tag{5.1c}$$

Equation (5.1b) implies that $\Psi_4(u,v)=0$ for $v<0$. Therefore, no gravitational radiation is incident from the right in $R_{\rm III}$ and the model represents the collision of an impulsive shell of null dust with an impulsive gravitational wave incident from the left. This observation together with the fact that no Coulomb-like field accompanies the interaction suggests that this particular model corresponds to the solution obtained by Babala recently.[20] Indeed, a rotation of the x^2-x^3 axes by an angle

$$\varphi=\tfrac{1}{2}\tan^{-1}(2q/(1-q^2)), \tag{5.2}$$

followed by the transformation

$$u\to u'=\begin{cases}1-2\sqrt{1-u}, & u\geqslant 0\\ u-1 & u\leqslant 0\end{cases} \tag{5.3}$$

brings the metric (4.4) corresponding to the present subcase to the form given in Ref. 20.

A-2: $a=1/2$. Now,

$$\Psi_0(u,v)=\frac{3(1+q^2)}{(1-v^2)A^2(A^2+16q^2v^2)^{1/2}}\{[(1-v)^5-6q^2(1-v^2)^2+q^4(1+v)^5+i2q(1-v^2)[(1+q^2)(5+v^2)v-2(1-q^2)(1+2v^2)]\}H(v)+(1+q^2)^{-1}[(1-q^2)+i2q]\delta(v),\quad u<0, \tag{5.4a}$$

where

$$A\equiv(1-v)^2+q^2(1+v)^2, \tag{5.4b}$$

$$\Psi_4(u,v)=\tfrac{1}{2}\delta(u),\quad v<0, \tag{5.5}$$

and

$$\Psi_2(u,v)=[G(S,v)/2t^2SA^2]H(u)H(v), \tag{5.6a}$$

where

$$G(x,y)\equiv(x-y)^4(x^2+xy+y^2)-6q^2xy(x^2-y^2)^2-q^4(x+y)^4(x^2-xy+y^2)-i2q(1+q^2)(x^2-y^2)^3. \tag{5.6b}$$

Equations (5.4) and (5.5) show that in the present model a variably polarized wave of the shock + impulsive variety collides with an impulsive wave accompanying an infinitely thin shell of null dust, which is incident from the opposite direction. Equation (5.6), on the other hand, shows that, as a result of the above collision, a curvature singularity develops as one approaches the hypersurface $t=0$ from the interior of the region of interaction.

A model representing the collision of an impulsive shell of null dust with a variably polarized gravitational plane wave was first constructed by Feinstein and Senovilla[14] quite recently. The solution technique used by the above authors, as well as the detailed features of their model, differ from ours. Both solutions, however, share the characteristic of ending up into a space-time singularity.

A-3: $a=1$. One can easily verify that, when the value of the parameter a becomes unity, the character of the gravitational wave incident from the left remains the same as in the previous subcase. However, the gravitational wave accompanying the null dust shell incident from the right becomes one of the shock + impulsive kind, too, but having constant polarization. Specifically,

$$\Psi_4(u,v)=\tfrac{3}{2}e^{2U}H(u)+\delta(u),\quad v<0. \tag{5.7}$$

The evolution of this model is completely different from the one observed in the two preceding subcases: No curvature singularity develops along the hypersurface $t=0$ and the solution can be continued analytically across it. This result follows from the observation that $\Psi_2(u,v)$ remains bounded as $t\to 0^+$, since

$$\Psi_2^{IV}(u,v)=6I(S,v)/ST^2, \tag{5.8a}$$

where

$$I(x,y)\equiv(x-y)^6-12q^2xy(x^2-y^2)^2-q^4(x+y)^6$$
$$-i2q[(x-y)^4(y^2+4xy+x^2)$$
$$+q^2(x+y)^4(y^2-4xy+x^2)], \tag{5.8b}$$

$$T\equiv(S-v)^4+q^2(S+v)^4. \tag{5.8c}$$

B. Case B: $n=m=1$ ($\delta_1=\delta_2=1/2$)

It turns out that this subclass of our models belongs to the family of solutions obtained by Ernst, Garcia, and Hauser[9–11] recently. For comparison, we give explicitly the metric coefficients in the region of interaction:

$$\chi=[(1-\eta^2)(1-\mu^2)]^a(A/B), \tag{5.9a}$$

$$q_2=-2q[(1-\eta^2)(1-\mu^2)]^{2a}(\eta-\mu)/B, \tag{5.9b}$$

$$e^{-M}=\frac{[(1-\eta^2)(1-\mu^2)]^{a^2-1/4}}{(1+\eta)^{2a}(1+\mu)^{2a}RS}A, \tag{5.9c}$$

where, now,

$$A\equiv(1-\eta^2)(1-\eta)^{2a}(1+\mu)^{2a}$$
$$+q^2(1-\mu^2)(1+\eta)^{2a}(1-\mu)^{2a},$$
$$B\equiv(1+\eta)^2(1-\eta)^{2a}(1+\mu)^{2a} \tag{5.10}$$
$$+q^2(1+\mu)^2(1+\eta)^{2a}(1-\mu)^{2a}.$$

The following are the two most significant members of the $n=m=1$ subclass of solutions.

B-1: $a=0$. For this model

$$\Psi_0(u,v)=\{[(1-q^2)-i2q]/(1+q^2)\}\delta(v),\quad u<0, \tag{5.11}$$

$$\Psi_4(u,v)=\delta(u),\quad v<0. \tag{5.12}$$

Therefore, the model represents the collision of two impulsive gravitational plane waves with noncollinear polarization. In fact, it is the well-known Nutku–Halil[21] solution, studied in great detail recently by Chandrasekhar and Ferrari,[22] but written in a different coordinate system. Thus it is known that the above solution becomes singular along $t=0$.

B-2: $a=1/2$. In this case

$$\Psi_0(u,v)=\Psi_0^{II}(v)H(v)+\frac{(1-q^2)-i2q}{1+q^2}\delta(v),$$
$$u<0, \tag{5.13a}$$

where

$$\Psi_0^{II}(v)=\frac{3(v^2-1)}{(1+q^2)^{1/2}T^2(T^2+16q^2v^2)^{3/2}}\{[16q^4-(1-q^2)^4+(9-14q^2)v]-i2q(1+q^2)[(1+q^2)(5+v^2)v$$
$$-2(1-q^2)(1+2v^2)][(1+q^2)(1+6v^2+v^4)-4(1-q^2)(1+v^2)v]\}, \tag{5.13b}$$

with

$$T\equiv(1-v)^2+q^2(1+v)^2 \tag{5.13c}$$

and

$$\Psi_4(u,v)=3e^{2U}(1+u)H(u)+\delta(u),\quad v<0. \tag{5.14}$$

Therefore, the model represents the collision of the same impulsive waves involved in the Nutku–Halil[21] solution, but when both are accompanied by shock waves. Of the latter, the right-moving one is variably polarized while the left-moving one has constant polarization. The result of this superposition, however, is impressive: The new solution does not develop a singularity along $t=0$. This follows the fact that $\Psi_2^{IV}(u,v)$, given by

$$\Psi_2^{IV}(u,v)=[(1+\eta)^2(1+\mu)^2/RSA^2]\Big\{(1-\eta)^3-q^2(1-\mu)[2-(1-\eta)(\eta+\mu)+2\eta^2\mu-q^2(1-3\mu^2+2\mu^3)]$$
$$+\frac{iq(1+\eta)(1+\mu)}{B[(1-\eta)^3(1+\mu)+q^2(1-\mu)^3(1+\eta)]}[(1-\eta)^6(1+\eta)(1+\mu)(2+\eta-3\mu)+q^4(1-\eta)$$
$$\times(1-\mu)^3(6-5\eta+2\eta^2-7\eta^3-\mu-16\eta\mu+29\eta^2\mu+2\mu^2-15\eta\mu^2+6\eta^2\mu^2-5\eta^3\mu^2+5\eta^3$$
$$-\eta^2\mu^3)-q^6(1+\eta)(1+\mu)(1-\mu)^6(2-3\eta+\mu)]\Big\}, \quad (5.15)$$

remains bounded as $t\to 0^+$.

C. Case C: $n=2, m=1$ ($\delta_1=\delta_2=\sqrt{6}/4\equiv k$)

According to Eqs. (4.15) and (4.16), the wave incident from the right in all the models of this subclass has a shock front, since

$$\Psi_4^{III}(u)=4e^{2U}[k+9au^2+24ka^2u^4+2a(4a^2-1)u^6]H(u),\quad v<0. \quad (5.16)$$

The following two subcases are typical examples of the colliding waves models covered by this category.

C-1: $a=0$. In this case

$$\Psi_0(u,v)=\{[(1-q^2)-i2q]/(1+q^2)\}\delta(v),\quad u<0. \quad (5.17)$$

Therefore, the wave incident from the left is an impulsive one and the outcome of its collision with the shock wave incident from the right is the formation of a space-time singularity along $t=0$. This can be inferred from the fact that $\Psi_2^{IV}(u,v)$ grows beyond all bounds as $t\to 0^+$. The latter quantity is given by

$$\Psi_2^{IV}(u,v)=(4ku/t^2RSA^2B)\{B[(1-\eta^2)^{4k}(2-(1-k)\eta^2-(1+k)\mu^2)-6kq^2(\eta^2-\mu^2)t^{4k}-q^4(1-\mu^2)^{4k}$$
$$\times(2-(1-k)\mu^2-(1+k)\eta^2)]-iqt^{2k}(1-\eta^2)^{4k}[k^{-1}(\eta^2-\mu^2)(1+\eta)(1-\mu)$$
$$+(2+3\eta+6\eta^2+3\eta^3)-\mu(3+7\eta+\eta^2-\eta^3)-\mu^2(1+2\eta+5\eta^2)+2\mu^3(1+\eta)]\}, \quad (5.18a)$$

where, now,

$$A\equiv(1-\eta^2)^{2k}[(1-\eta)(1+\mu)]^{2a}+q^2(1-\mu^2)^{2k}[(1+\eta)(1-\mu)]^{2a},$$

$$B\equiv(1-\eta^2)^{2k}[(1-\eta)(1+\mu)]^{2a}(1+\eta)(1-\mu)+q^2(1-\mu^2)^{2k}[(1+\eta)(1-\mu)]^{2a}(1-\eta)\times(1+\mu). \quad (5.18b)$$

C-2: $a=1-k$. Now the impulsive wave incident from the left in the previous case is accompanied by a shock one, since

$$\Psi_0(u,v)=\Psi_0^{II}(v)H(v)+\frac{(1-q^2)-i2q}{(1+q^2)}\delta(v),\quad u<0, \quad (5.19a)$$

where

$$\Psi_0^{II}(v)=\frac{1}{2(1-v^2)A^2T}\{[O(v)-q^2P(v)-q^4\times P(-v)+q^6O(-v)]-i2q(1-v^2)^{1+2k}\times[Q(v)-2q^2Z(v)-q^4Q(-v)]\}, \quad (5.19b)$$

with

$$T\equiv[A^2+16q^2v^2(1-v^2)^{4(1+k)}]^{1/2},$$

$$O(x)\equiv(1+x)^{12k}(1-x)^{12}\{3(4+7x+2x^2)-2k[6+24x+19x^2-16kx(1+2x-kx)]\},$$

$$P(x)\equiv(1+x)^{8k+3}(1-x)^{4k+7}\{3(20+121x+22x^2+7x^3+6x^4)-2k[(30+360x+113x^2+24x^3+57x^4)-16kx(11+10x+x^2+6x^3)+16k^2x^2(5+3x^2)]\},$$

$$Q(x)\equiv(1+x)^{8k}(1-x)^8\{3(8+23x+4x^2-9x^3)-4k[(6+36x+19x^2-12x^3)-4kx(5+8x-x^2)+16k^2x^2]\},$$

$$Z(x)\equiv x(1-x^2)^{4(k+1)}\{3(41+9x^2)-16k(15+3x^2)+16k^2(7+x^2)\}. \quad (5.19c)$$

In the present case, however, no singularity forms along the $t=0$ hypersurface. This is reflected in the fact that the following expression for $\Psi_2^{\rm IV}(u,v)$ remains bounded as $t\to 0^+$.

$$\Psi_2^{\rm IV}(u,v)=[4u/RSA^2B(1+\eta)(1+\mu)]\{B[J(\eta,\mu)-6q^2K(\eta,\mu)-q^4J(\mu,\eta)]-i2q\times(1+\eta)^{2k+1}(1+\mu)^{2k+1}[L(\eta,\mu)+2q^2N(\eta,\mu)+q^4L(\mu,\eta)]\}, \quad (5.20a)$$

where

$$J(x,y)\equiv(1+y)^4(1-x)^3(1+x)^{8k}\{(1+x)(x+y)+k(1-x)(2+3x-y)-2k^2(1-x)\times(x-y)\},$$

$$K(x,y)\equiv(x-y)(1-x^2)(1-y^2)(1+x)^{4k+1}\times(1+y)^{4k+1}[(x+y)-2k(1-k)\times(1-x)(x-y)],$$

$$L(x,y)\equiv(1+y)^3(1-x)^3(1+x)^{8k}\{(1+2x-2y-xy)(x+y)+k(1-x)(1-y)\times[(2+5x-3y)+4k(x+y)]\},$$

$$N(x,y)\equiv(1-x^2)(1-y^2)(1+x)^{4k}(1+y)^{4k}\{(x+y)\times[(1+xy)^2-2(x^2+y^2)]+k(1-x)\times(x-y)[(1+x)(1+y)(2-x-y)-3(x-y)^2+4k(x-y)^2]\}. \quad (5.20b)$$

VI. SINGULARITIES

As shown by the analysis of the specific models considered in the last section, some members of our new family of solutions become singular along the hypersurface $t=0$, while others do not. The question which is exactly the subclass of solutions that does not develop a singularity along the "focusing hypersurface" $t=0$ could be answered by calculating the Weyl scalar $\Psi_2^{\rm IV}(u,v)$ for the whole class, using Eqs. (3.7), (4.10), and (4.13c). This would be a formidable task, however, as can be inferred from the complexity of the expressions for $\Psi_2^{\rm IV}$ corresponding to the particular examples already considered. Fortunately, an easier and more illuminating approach is now available thanks to Yurtsever's[23,24] recent work on this subject.

Specifically, Yurtsever[23,24] has shown that, as one approaches the $t=0$ hypersurface from the interior of the region of interaction, the metric tends to an "inhomogeneous Kasner solution." This means that as $t\to 0^+$,

$$ds^2\to(\epsilon_0\,d\tau)^2-(\epsilon_1\tau^{p_1}\,dz)^2-(\epsilon_2\tau^{p_2}\,dX^2)^2-(\epsilon_3\tau^{p_3}\,dX^3)^2, \quad (6.1)$$

where τ is a monotonic function of t, X^a are z-dependent linear combinations of x^2 and x^3, $\epsilon_i=\epsilon_i(z)$ and $p_i=p_i(z)$, with

$$p_1+p_2+p_3=p_1^{\,2}+p_2^{\,2}+p_3^{\,2}. \quad (6.2)$$

Therefore, the $t\to 0^+$ behavior of the curvature tensor of a given solution representing the collision of gravitational plane waves can be determined from that of its Kasner asymptote, in the sense of Eq. (6.1). Moreover, by finding the Kasner asymptote of the given metric, one can determine the "fine structure" of the singularity formation process, since the Kasner exponents $\{p_i\}$ specify the directions in which the gravitational rays get focused and those along which defocusing occurs.

We consider the collinear limit of our solutions first. It is obtained by letting $q\to 0$ in Eqs. (3.7), whereby one ends up with the seed metric given by Eqs. (3.1), but with (δ_1,δ_2) replaced by $(\delta_1+1/2,\delta_2-1/2)$. One easily finds that the Kasner exponents in the $t\to 0^+$ limit of our collinear models are given by

$$p_1=\frac{\alpha(\alpha+1)}{\beta},\quad p_2=\frac{(\alpha+1)}{\beta},\quad p_3=-\frac{\alpha}{\beta},\tag{6.3}$$

where

$$\alpha\equiv a+\delta_1 \text{ and } \beta\equiv\alpha^2+\alpha+1.\tag{6.4}$$

Moreover, $\tau = t^{\beta}$ and $(X^2,X^3) = (x^2,x^3)$ in the present case. As shown by Yurtsever,[23,24] all models in which $p_1\neq 0$ develop a curvature singularity along the focusing hypersurface $t=0$. Therefore, all of our collinear solutions become singular as $t \to 0^+$, unless $\alpha\in\{-1, 0\}$. In the latter case the curvature singularity along $t=0$ gives its place to a Killing–Cauchy horizon across which the metric can be continued analytically.

The above result regarding the singularity behavior of the collinear models was already proved by the present authors in Refs. 16 and 17, using the Weyl scalars approach. Now we can add the following observations. When $\alpha < -1$, then $p_1, p_3 > 0$ and $p_2 < 0$. Therefore, in the subclass of the diagonal solutions in which $\alpha < -1$ one observes unbounded contraction of physical volumes in the directions of the z and x^3 axes and unbounded expansion along the x^2 axis on approaching the singularity hypersurface $t=0$. The corresponding singularity is called astigmatic, because the rays of the incident gravitational waves get focused along one of the transverse directions, namely, x^3, and defocused along the other, namely, x^2. When $\alpha\in(-1, 0)$, on the other hand, then $p_1 < 0$ and $p_2, p_3 > 0$. The singularity that now develops as $t \to 0^+$ is anastigmatic and is accompanied by an unbounded expansion of the physical volumes in the direction of the waves' propagation and unbounded contraction in the transverse directions. Finally, when $\alpha > 0$, then the process of singularity formation is the same as when $\alpha < -1$, but with the roles of the x^2 and x^3 axes interchanged.

Turning to the $q\neq 0$ models, corresponding to the collision of noncollinearly polarized waves, we find the following singularity behavior.

(i) When $\alpha < -1/2$, the nondiagonal models evolve exactly as the diagonal ones described above. This shows that the difference in polarization of the incoming waves has no effect on the formation of a singularity along $t=0$, as long as $\alpha < -1/2$.

(ii) For $\alpha\in[1/2, 0)$, the situation is similar as in the previous case, except for the fact that the Kasner axes X^2, X^3 are rotated relative to x^2 and x^3 and the coefficients ϵ_μ are z dependent.

(iii) When $\alpha\in[0, 1/2]$, the Kasner exponents become

$$p_1=\frac{\alpha(\alpha-1)}{\beta'},\quad p_2=\frac{\alpha}{\beta'},\quad p_3=\frac{1-\alpha}{\beta'},\tag{6.5}$$

where

$$\beta'\equiv\alpha^2-\alpha+1.\tag{6.6}$$

Moreover, $\tau = t^{\beta'}$ in this case, while the ϵ_μ's and the X^a's are z dependent as in the previous interval of values of α. From Eq. (6.5) it follows that the $\alpha=0$ nondiagonal models remain free of singularities along $t=0$, like their diagonal counterparts. However, the singular models (corresponding to $\alpha\neq 0$) behave quite differently from the diagonal ones, in the sense that the astigmatic singularity of the latter gives its place to an anastigmatic one. This shows that the difference in polarization starts having a considerable effect as soon as α enters the positive side of the real axis. This is brought out even more clearly by the following case.

(iv) When $\alpha > 1/2$, the results of case (iii) still hold and, therefore, no singularity appears on the $t=0$ hypersurface when $p_1 = 0$, i.e., when $\alpha = 1$. This is in complete contrast with the behavior of the diagonal models which are always singular along $t=0$ for all $\alpha > 0$.

Let us point out that all three cases A-3, B-2, and C-2 considered in Sec. V provide concrete examples of the models comprising category (iv). This fact can also be deduced from the explicit expressions for $\Psi_2^{\rm IV}(u,v)$ corresponding to the three models above, which are given by

$$\Psi_2^{\rm IV}(u,v)=6(S+v)^2/t^2,\tag{6.7}$$

$$\Psi_2^{\rm IV}(u,v)=\frac{(1+\eta)(1-\mu^2)}{SRt^2},\tag{6.8}$$

and

$$\Psi_2^{\rm IV}(u,v)=[4u(1-\mu)/SRt^2]\{(1+\eta)(\eta+\mu)+k(1-\eta)(2+3\eta-\mu)-2k^2(1-\eta)(\eta-\mu)\},\tag{6.9}$$

for $q=0$, respectively.

The significant role played by the difference in polarization in the outcome of the head-on collision of gravitational plane waves which was exhibited by the models obtained above is not surprising, of course. It must be noted, however, that, to our knowledge, this is the first case where the polarization of one of the incident waves can be controlled freely, thus allowing for the explicit expression of the influence of polarization on the result of the interaction.

[1] V. A. Belinsky and V. E. Zakharov, Sov. Phys. JETP **48**, 985 (1978).
[2] V. A. Belinsky and V. E. Zakharov, Sov. Phys. JETP **50**, 1 (1979).
[3] P. Szekeres, Nature (London) **228**, 1183 (1970).
[4] P. Szekeres, J. Math. Phys. **13**, 286 (1972).
[5] K. A. Khan and R. Penrose, Nature (London) **229**, 185 (1971).
[6] V. Ferrari, in *General Relativity and Gravitation*, edited by N. Ashby, D. F. Bartlett, and W. Wyss (Cambridge U.P., Cambridge, England, 1990).

[7] V. Ferrari, J. Ibanez, and M. Bruni, Phys. Lett. A **122**, 459 (1987).
[8] V. Ferrari, J. Ibanez, and M. Bruni, Phys. Rev. D **36**, 1053 (1987).
[9] F. J. Ernst, A. D. Garcia, and I. Hauser, J. Math. Phys. **28**, 2155 (1987).
[10] F. J. Ernst, A. D. Garcia, and I. Hauser, J. Math. Phys. **28**, 2951 (1987).
[11] F. J. Ernst, A. D. Garcia, and I. Hauser, J. Math. Phys. **29**, 681 (1988) and references therein.
[12] W. Li, I. Hauser, and F. J. Ernst, J. Math. Phys. **32**, 723 (1991).
[13] W. Li, I. Hauser, and F. J. Ernst, J. Math. Phys. **32**, 1025 (1991).
[14] A. Feinstein and J. M. M. Senovilla, Phys. Lett. A **138**, 102 (1989).
[15] A. Economou and D. Tsoubelis, J. Math. Phys. **30**, 1562 (1989).
[16] D. Tsoubelis and A. Z. Wang, Gen. Relat. Grav. **21**, 807 (1989).
[17] D. Tsoubelis and A. Z. Wang, Gen. Relat. Grav. **22**, 1091 (1990).
[18] P. Szekeres, J. Math. Phys. **6**, 1387 (1965).
[19] P. Szekeres, J. Math. Phys. **7**, 751 (1966).
[20] D. Babala, Class. Quantum Grav. **4**, L89 (1987).
[21] Y. Nutku and M. Halil, Phys. Rev. Lett. **39**, 1379 (1977).
[22] S. Chandrasekhar and V. Ferrari, Proc. R. Soc. Lond. Ser. A **396**, 55 (1984).
[23] U. Yurtsever, Phys. Rev. D **38**, 1706 (1988).
[24] U. Yurtsever, Phys. Rev. D **40**, 329 (1989).